Arrays: computing with many numbers

Some perspective

- We have so far (mostly) looked at what we can do with single numbers (and functions that return single numbers).
- Things can get much more interesting once we allow not just one, but many numbers together.
- It is natural to view an array of numbers as one object with its own rules.
- The simplest such set of rules is that of a **vector.**

Demo "Sound as Vectors"

Vectors

A vector is an element of a Vector Space

$$\boldsymbol{n}\text{-vector:} \qquad \boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = [x_1 \quad x_2 \cdots x_n]^T$$

Vector space $\mathcal{V}$:

A vector space is a set $\mathcal{V}$ of vectors and a field $\mathcal{F}$ of scalars with two operations:

1) addition: $u + v \in \mathcal{V}$, and $u, v \in \mathcal{V}$

2) multiplication : $\alpha \cdot u \in \mathcal{V}$, and $u \in \mathcal{V}$, $\alpha \in \mathcal{F}$

Vector Space

The addition and multiplication operations must satisfy:

(for $\alpha, \beta \in \mathcal{F}$ and $u, v \in \mathcal{V}$)

Associativity: $u + (v + w) = (u + v) + w$

Commutativity: $u + v = v + u$

Additive identity: $v + 0 = v$

Additive inverse: $v + (-v) = 0$

Associativity wrt scalar multiplication: $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v$

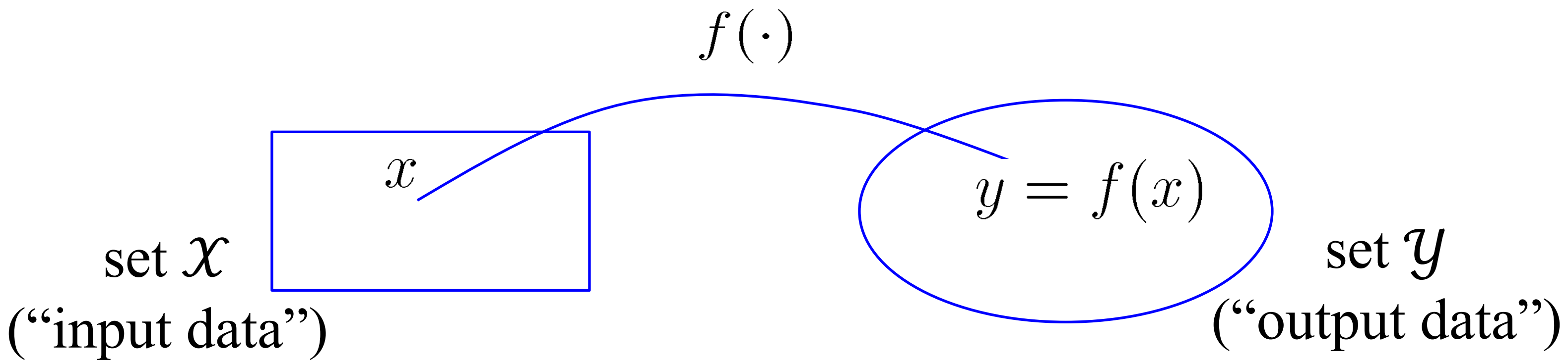
Distributive wrt scalar addition: $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$

Distributive wrt vector addition: $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$

Scalar multiplication identity: $1 \cdot (u) = u$

Linear Functions

Function: $f : \mathcal{X} \to \mathcal{Y}$



The function f takes vectors $\boldsymbol{x} \in \mathcal{X}$ and transforms into vectors $\boldsymbol{y} \in \mathcal{Y}$

A function f is a linear function if

(1) $f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$

(2) $f(a\boldsymbol{u}) = a\,f(\boldsymbol{u})$ for any scalar a

Iclicker question

1) Is

$$f(x) = \frac{|x|}{x},\ f: \mathcal{R} \to \mathcal{R}$$

a linear function?

A) YES

B) NO

2) Is

$$f(x) = a\,x + b,\ f: \mathcal{R} \to \mathcal{R},\ \ a, b \in \mathcal{R} \text{ and } a, b \neq 0$$

a linear function?

A) YES

B) NO

Matrices

- $m \times n$-matrix $\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}$

- Linear functions $f(\boldsymbol{x})$ can be represented by a Matrix-Vector multiplication.

- Think of a matrix $\boldsymbol{A}$ as a linear function that takes vectors $\boldsymbol{x}$ and transforms them into vectors $\boldsymbol{y}$

$$\boldsymbol{y} = f(\boldsymbol{x}) \rightarrow \boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$$

- Hence we have:

$$\boldsymbol{A}\,(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{A}\,\boldsymbol{u} + \boldsymbol{A}\,\boldsymbol{v}$$
$$\boldsymbol{A}\,(\alpha\,\boldsymbol{u}) = \alpha\,\boldsymbol{A}\,\boldsymbol{u}$$

Matrix-Vector multiplication

- Recall summation notation for matrix-vector multiplication $\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$

- You can think about matrix-vector multiplication as:

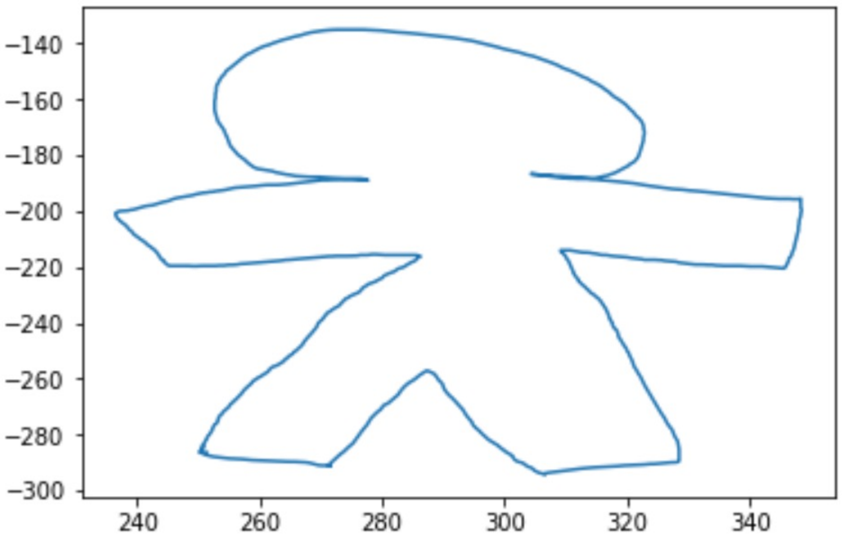
Linear combination of column vectors of **A**

$$\boldsymbol{y} = x_1\,\mathbf{A}[:,1] + x_2\,\mathbf{A}[:,2] + \cdots + x_n\,\mathbf{A}[:,n]$$

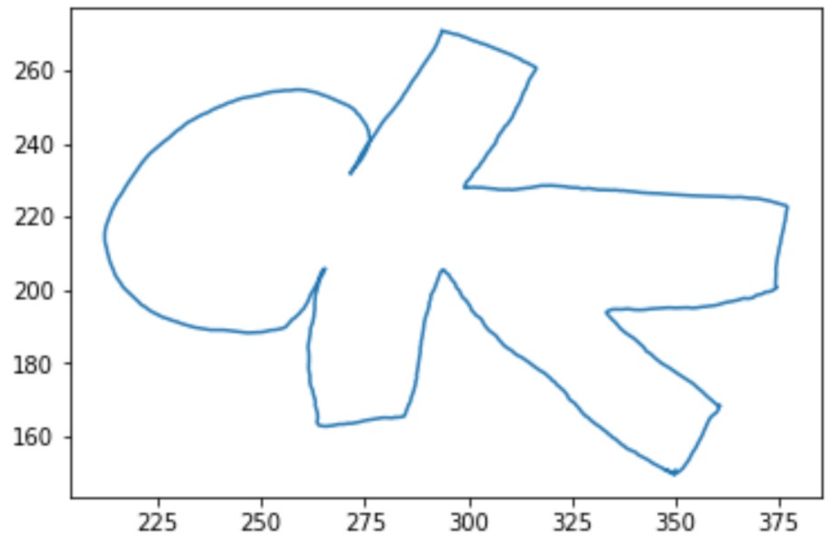
Dot product of $\boldsymbol{x}$ with rows of **A**

$$\boldsymbol{y} = \begin{pmatrix} \mathbf{A}[1,:]\cdot\boldsymbol{x} \\ \vdots \\ \mathbf{A}[m,:]\cdot\boldsymbol{x} \end{pmatrix}$$

Matrices operating on data



Data set: $\boldsymbol{x}$



Data set: $\boldsymbol{y}$

Rotation

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$$

or

$$\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$$

Example: Shear operator

Matrix-vector multiplication for each vector (point representation in 2D):

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.25 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 1 \end{pmatrix}$$

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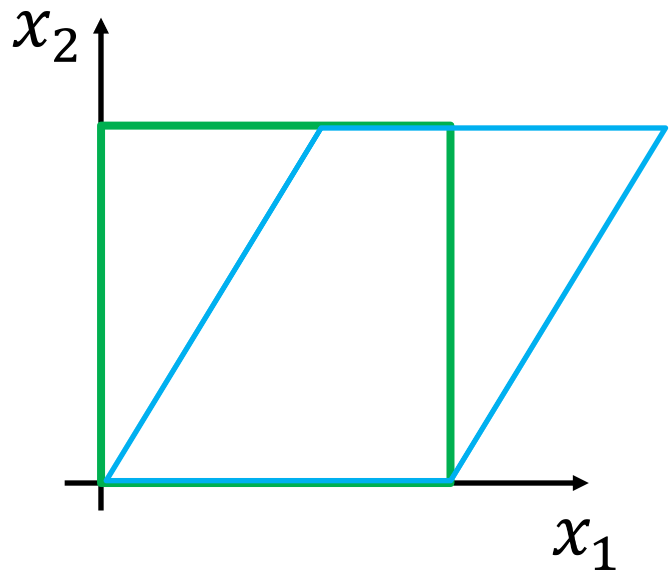


We can do this better…

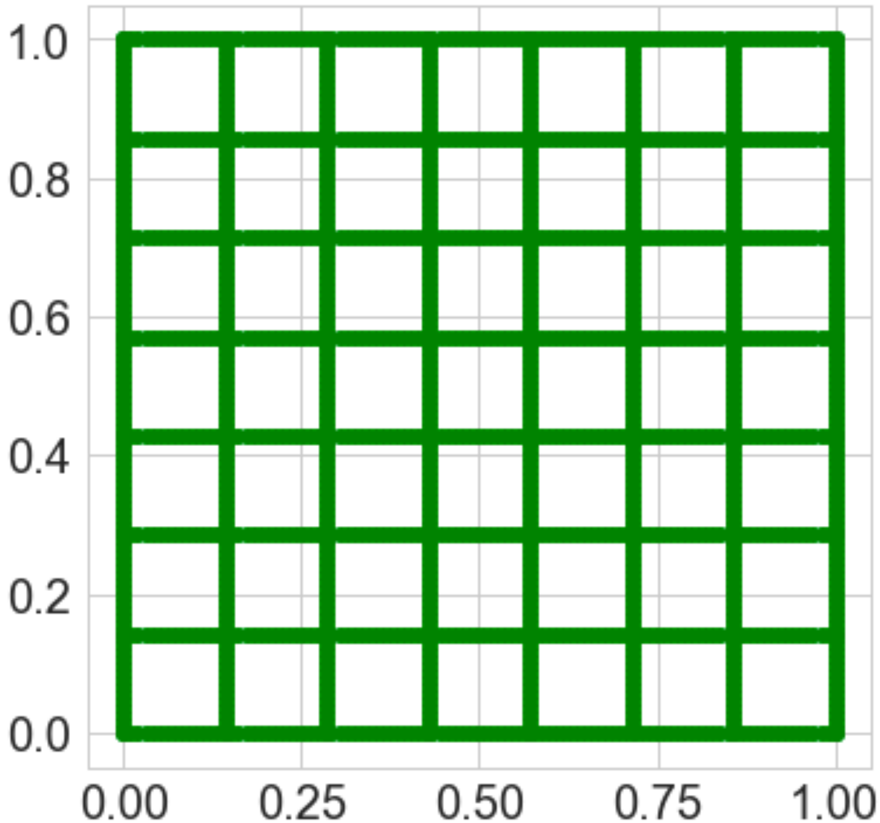
$$\begin{pmatrix} y_{11} & \dots & y_{1n} \\ y_{21} & \dots & y_{2n} \end{pmatrix} = \begin{pmatrix} 1 & 0.25 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & \dots & x_{1n} \\ x_{21} & \dots & x_{2n} \end{pmatrix}$$

$$\underset{(2\times n)}{\boldsymbol{Y}} = \underset{(2\times 2)}{\boldsymbol{A}}\ \underset{(2\times n)}{\boldsymbol{X}}$$

n: number of data points in the set

Matrices as operators

- **Data**: grid of 2D points
- Transform the data using matrix multiply



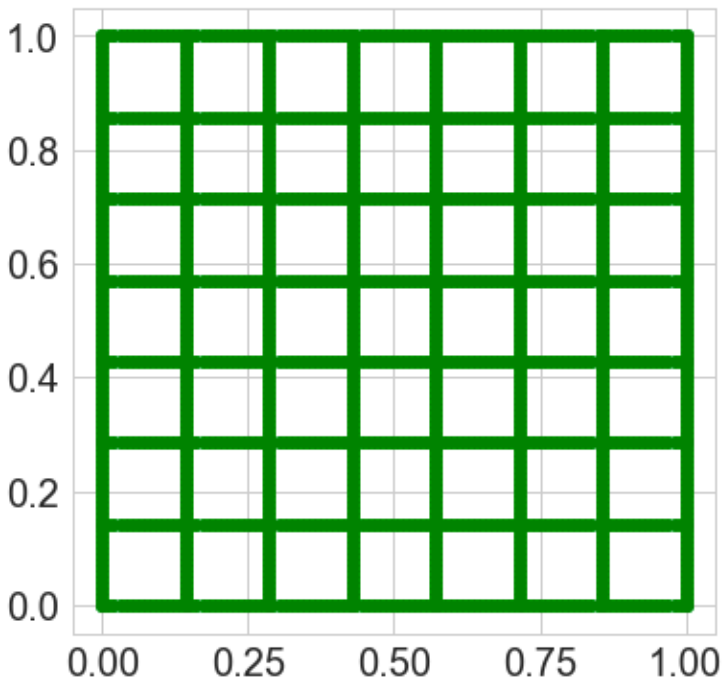
What can matrices do?

1. Shear
2. Rotate
3. Scale
4. Reflect

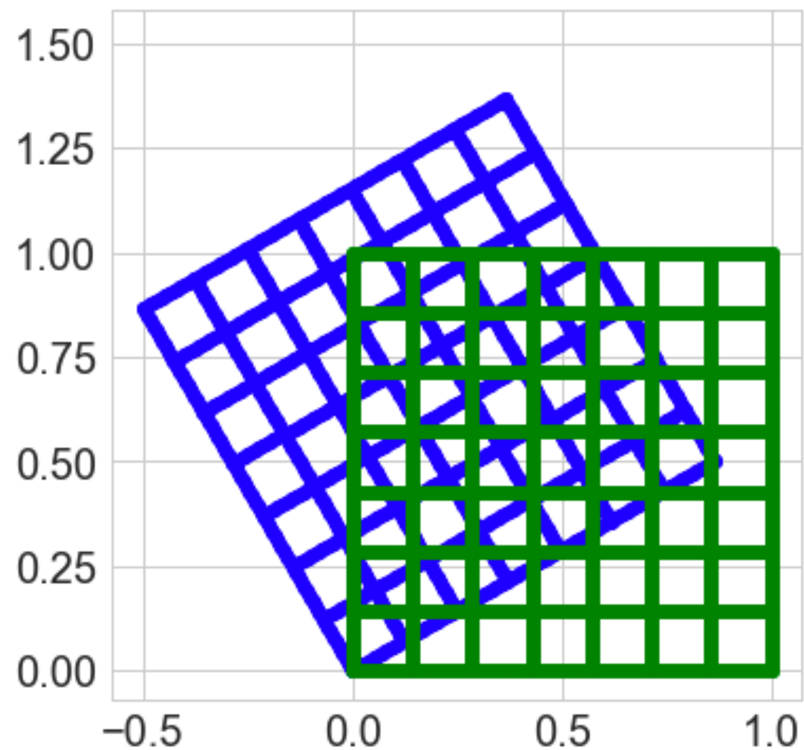
5. Can they translate?

Rotation operator

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

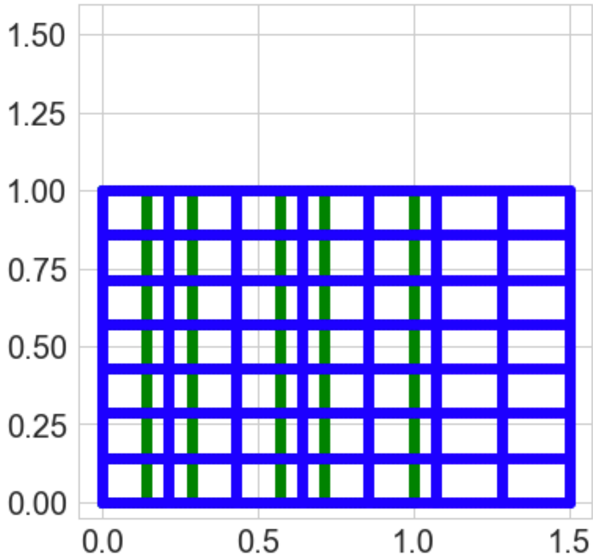


$\theta = \pi/6$



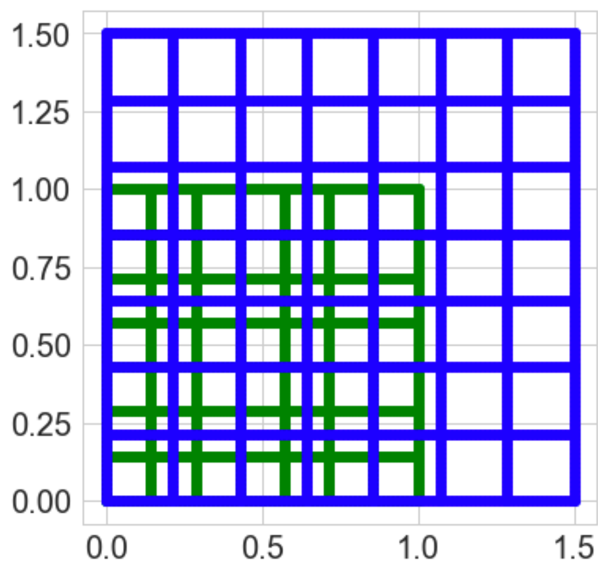
Scale operator

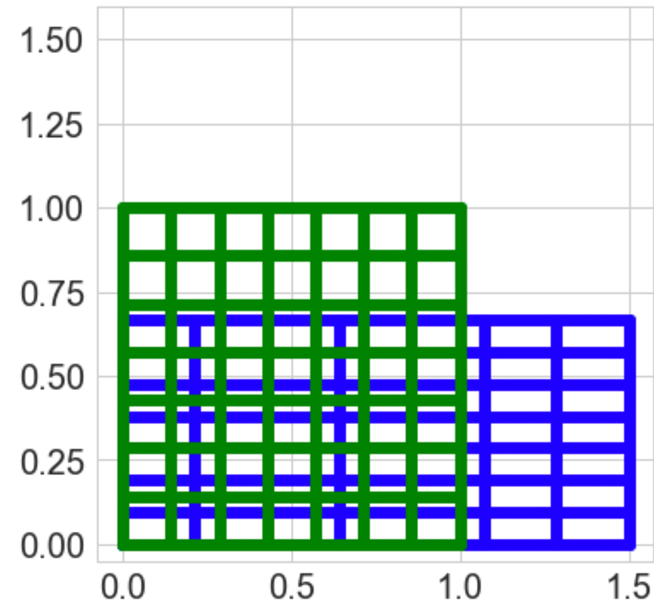
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$$\begin{pmatrix} 3/2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$



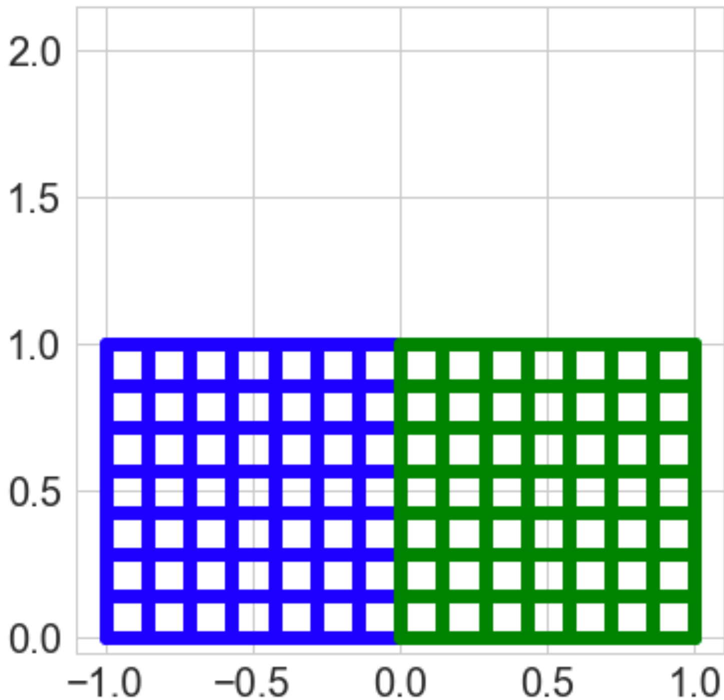


$$\begin{pmatrix} 3/2 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Reflection operator

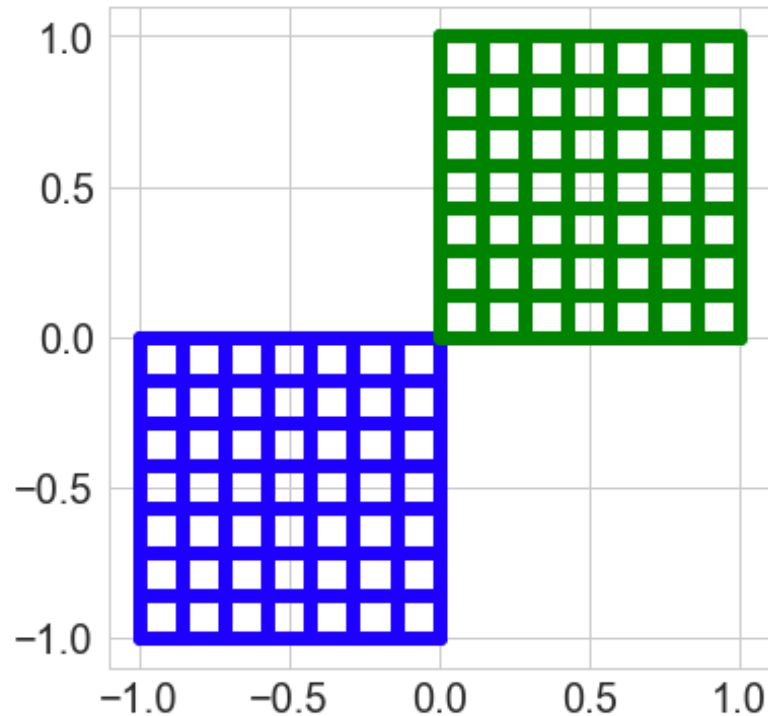
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Reflect about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

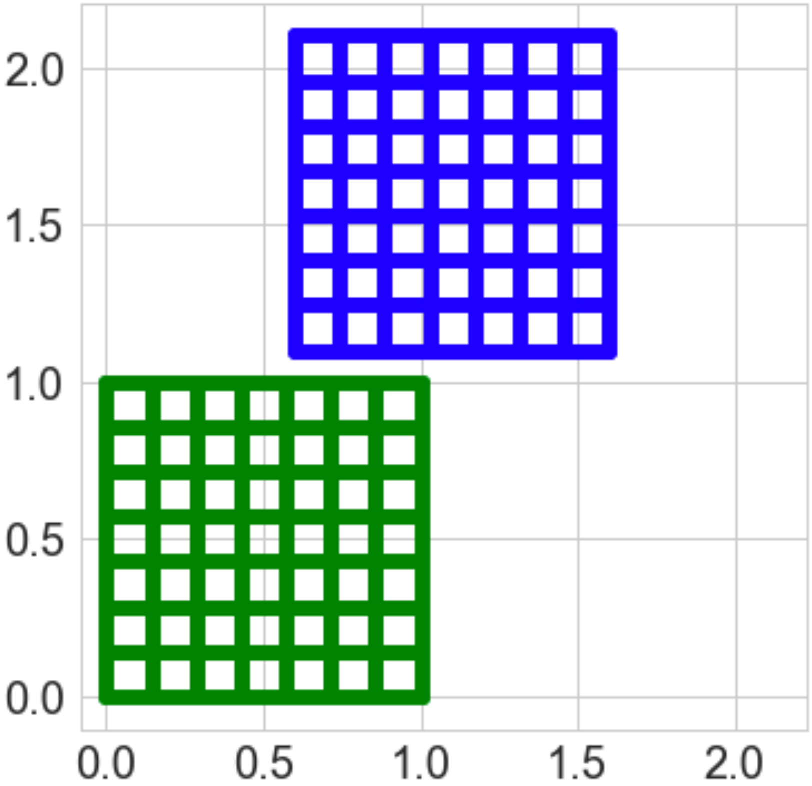


Reflect about x and y-axis

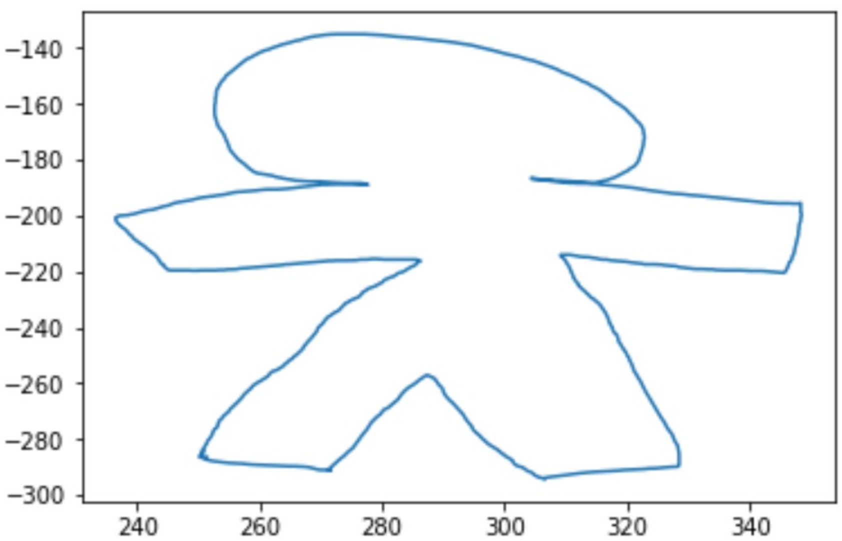
Translation (shift)

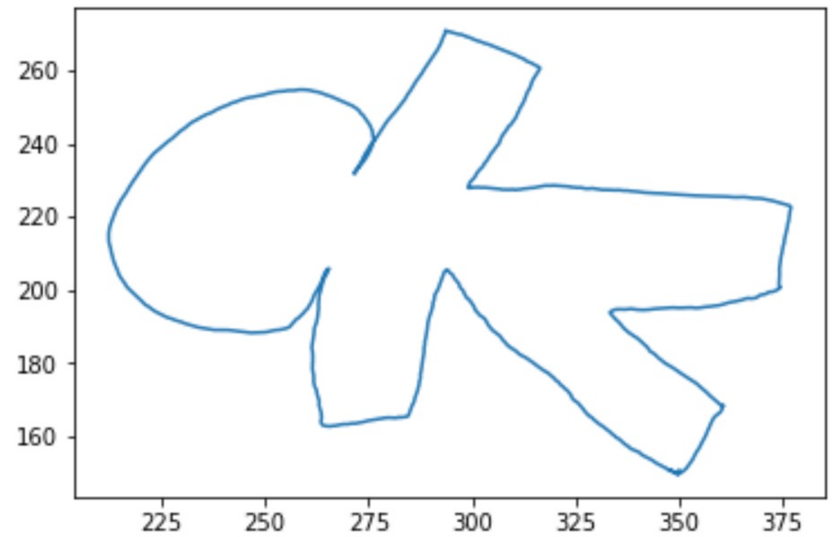
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$a = 0.6; b = 1.1$



Matrices operating on data





Data set: ***A***

Data set: ***B***

Rotation

Demo "Matrices for geometry transformation"

Iclicker question

A triangle has vertices with coordinates (1,1), (2,1) and (2,4). The triangle is transformed by the matrix

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 0.75 \end{pmatrix}$$

What is the operation defined by the matrix above?

a) Expand
b) Shrink
c) Scale
d) Rotate
e) Reflect

What are the coordinates of the transformed triangle?

a) (0.75,1.5), (0.75,3), (3,3)
b) (1.5,0.75), (3,0.75), (3,3)
c) (1.5,0.75), (1.5,1.5), (6,1.5)
d) (0.75,1.5), (1.5,1.5), (1.5,6)

Notation and special matrices

- Square matrix: $m = n$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Zero matrix: $A_{ij} = 0$
- Identity matrix $[\boldsymbol{I}] = [\delta_{ij}]$
- Symmetric matrix: $A_{ij} = A_{ji}$ $[\boldsymbol{A}] = [\boldsymbol{A}]^T$
- Permutation matrix:
  - Permutation of the identity matrix
  - Permutes (swaps) rows

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ a \\ b \end{pmatrix}$$

- Diagonal matrix: $A_{ij} = 0,\ \forall i, j \mid i \neq j$
- Triangular matrix:

Lower triangular: $L_{ij} = \begin{cases} L_{ij}, i \geq j \\ 0, \quad i < j \end{cases}$

Upper triangular: $U_{ij} = \begin{cases} U_{ij}, i \leq j \\ 0, \quad i > j \end{cases}$

More about matrices

- Rank: the rank of a matrix $\boldsymbol{A}$ is the dimension of the vector space generated by its columns, which is equivalent to the number of linearly independent columns of the matrix.
- Suppose $\boldsymbol{A}$ has shape $m \times n$:
  - $rank(\boldsymbol{A}) \leq \min(m, n)$
  - Matrix $\boldsymbol{A}$ is **full rank**: $rank(\boldsymbol{A}) = \min(m, n)$. Otherwise, matrix $\boldsymbol{A}$ is **rank deficient**.
- Singular matrix: a square matrix $\boldsymbol{A}$ is invertible if there exists a square matrix **B** such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$. If the matrix is not invertible, it is called singular.

Norms

What's a norm?

- A generalization of 'absolute value' to vectors.
- $f(x) : \mathbb{R}^n \to \mathbb{R}_0^+$, returns a 'magnitude' of the input vector
- In symbols: Often written $\|\mathbf{x}\|$.

Define norm.

A function $\|\mathbf{x}\| : \mathbb{R}^n \to \mathbb{R}_0^+$ is called a norm if and only if

1. $\|\mathbf{x}\| > 0 \Leftrightarrow \mathbf{x} \neq \mathbf{0}$.
2. $\|\gamma\mathbf{x}\| = |\gamma| \, \|\mathbf{x}\|$ for all scalars γ.
3. Obeys triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$

Example of Norms

What are some examples of norms?

The so-called *p-norms*:

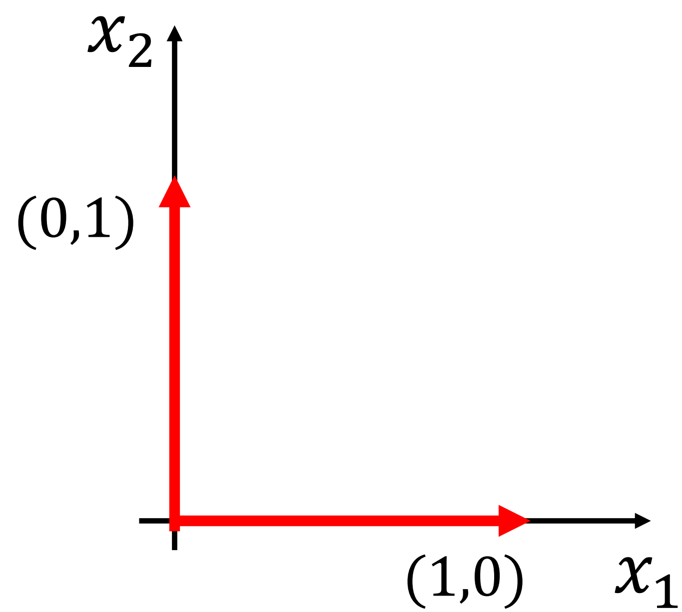
$$\left\| \begin{pmatrix} x_1 \\ \\ x_n \end{pmatrix} \right\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p} \quad (p \geqslant 1)$$

$p = 1, 2, \infty$ particularly important

Unit Ball: Set of vectors $\boldsymbol{x}$ with norm $\|\boldsymbol{x}\| = 1$

Demo "Vector Norms"

- Why should be $p \geq 1$ when calculating the p-norm?



$||(1,0) + (0,1)|| = ||(1,1)|| = 2^{1/p} \geq 2 = 1 + 1 = ||(1,0)|| + ||(0,1)||$ where in the critical step we use the fact that p is less than one. So we have a counterexample to the triangle inequality.

Basically, the main reason these are not norms is that the unit ball is not convex, which means you can pick two points in the unit ball, like (0,1) and (1,0), and draw a line between them and have points on that line with norm greater than one: for example $||(1/2, 1/2)|| = (2/2^p)^{1/p} = 2^{(1-p)/p}$ which is always greater than one if the exponent is positive, (which it is if $0 < p < 1$).

Iclicker question

Measure of Length vs. Measure of Distance

A norm can be used to measure the length of a vector x by calculating $||x||$. To calculate the distance between two points given by position vectors x and y, one would use:

Choice*

a) $||x|| + ||y||$

b) $||x|| - ||y||$

c) $||x + y||$

d) $|\,||x|| - ||y||\,|$

e) $||x - y||$

Norms and Errors

If we’re computing a vector result, the error is a vector.
That’s not a very useful answer to ‘how big is the error’.
What can we do?

Apply a norm!

How? Attempt 1:

$$\text{Magnitude of error} \neq \|\text{true value}\| - \|\text{approximate value}\|$$ **WRONG!**

Attempt 2:

$$\text{Magnitude of error} = \|\text{true value} - \text{approximate value}\|$$

Absolute and Relative Errors

What are the absolute and relative errors in approximating the location of Siebel center (40.114, −88.224) as (40, −88) using the 2-norm?

Absolute error:

a) *0.2240*

b) *0.3380*

c) *0.2513*

Relative error:

a) 2.59×10^{-3}

b) 2.81×10^{-3}

Matrix Norms

What norms would we apply to matrices?

- Easy answer: '*Flatten*' matrix as vector, use vector norm. This corresponds to an entrywise matrix norm called the Frobenius norm,

$$\|A\|_F := \sqrt{\sum_{i,j} a_{ij}^2}.$$

Matrix Norms

However, interpreting matrices as linear functions, what we are really interested in is the maximum amplification of the norm of any vector multiplied by the matrix,

$$||A|| := \max_{||\mathbf{x}||=1} ||A\mathbf{x}|| \,.$$

These are called induced matrix norms, as each is associated with a specific vector norm $||\cdot||$.

Matrix Norms

The following are equivalent:

$$\max_{\|x\|\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|x\|\neq 0} \left\| A \underbrace{\frac{\mathbf{x}}{\|\mathbf{x}\|}}_{y} \right\| \overset{\|\mathbf{y}\|=1}{=} \max_{\|\mathbf{y}\|=1} \|A\mathbf{y}\| = \|A\| .$$

Logically, for each vector norm, we get a different matrix norm, so that, e.g. for the vector 2-norm $\|\mathbf{x}\|_2$ we get a matrix 2-norm $\|A\|_2$, and for the vector ∞-norm $\|\mathbf{x}\|_\infty$ we get a matrix ∞-norm $\|A\|_\infty$.

Induced Matrix Norms

Given the matrix $\boldsymbol{A}$ with m rows and n columns:

$$\|\boldsymbol{A}\|_1 = \max_j \sum_{i=1}^{m} |A_{ij}|$$

Maximum absolute column sum of the matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_\infty = \max_i \sum_{j=1}^{n} |A_{ij}|$$

Maximum absolute row sum of the matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_2 = \max_k \sigma_k$$

σ_k are the singular value of the matrix $\boldsymbol{A}$

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

But also some more properties that stem from our definition:

1. $\|A\mathbf{x}\| \leqslant \|A\| \, \|\mathbf{x}\|$
2. $\|AB\| \leqslant \|A\| \, \|B\|$ (easy consequence)

Both of these are called submultiplicativity of the matrix norm.

Demo "Matrix Norms"

Iclicker question

Determine the norm of the following matrices:

1) $$\left\| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\|_\infty$$

2) $$\left\| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\|_1$$

a) 3

b) 4

c) 5

d) 6

e) 7

Iclicker question

Matrix Norm Approximation

Suppose you know that for a given matrix A three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ for the vector norm $||\cdot||$,

$||\mathbf{x}|| = 2$, $||\mathbf{y}|| = 1$, $||\mathbf{z}|| = 3$,

and for corresponding induced matrix norm,

$||A\mathbf{x}|| = 20$, $||A\mathbf{y}|| = 5$, $||A\mathbf{z}|| = 90$.

What is the largest lower bound for $||A||$ that you can derive from these values?

a) 90
b) 30
c) 20
d) 10
e) 5

Induced Matrix Norm of a Diagonal Matrix

What is the 2-norm-based matrix norm of the diagonal matrix

$$A = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}?$$

Induced Matrix Norm of an Inverted Diagonal Matrix

What is the 2-norm-based matrix norm of the **inverse** of the diagonal matrix

$$A = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}?$$